

Quantum Gravity in $D = 5$ Dimensions

by

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and

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Abstract

We propose a topological Chern-Simons term in $D = 5$ dimensions coupled to Einstein Hilbert theory. Hartree approximation for topological Lagrangian and the Chern-Simons term in $D = 3$ is considered. An effective model of Quantum Gravity in $D = 5$ dimensions is presented here. The analysis of residues is considered and the unitarity is guaranteed at tree level. The propagator is ghost and tachyon free.

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Introduction

It's known that the theory of Quantum Gravity has numerous problems. In particular the perturbative approach contains the insoluble conflict between unitarity and renormalizability in $D = 4$ dimensions. The Einstein Hilbert theory, for example, is unitarity but is not renormalizable. Anyway the theory can be at least seen as an effective theory. Many models for perturbative quantum gravity have been constructed in $D = 4$ including theories with higher derivatives[4]. Another interesting development occurred with the discovery of topological Chern-Simons term. That term despite having some important geometric properties doesn't have any physics associated with it. The Chern-Simons term in $D = 3$ dimensions together with Einstein-Hilbert Lagrangian provides a rich array of physics. All problems of Quantum Gravity in $D = 4$ disappear in $D = 3$. The so called topological theory for gravitation or still the Einstein-Hilbert-Chern-Simons theory is unitarity and finite[2, 4].

It can be shown that the dynamics is given by a massive pole and the massless pole doesn't have propagation in $D = 3$ [4].

Presently we know that in general it is possible to have a topological term like Chern-Simons in odd dimensions[6], $D = 3, 5, 7 \dots$.

The question, however is what we might do to make a perturbative attack on Einstein-Chern-Simons in $D = 3$, but may we do the same approach for $D = 5$ dimensions?

Is it possible to carry out a perturbative approach in 5-dimensions as was the case for $D = 3$?

To answer this question a convenient "Chern-Simons" in $D = 5$ is needed and then to establish that a perturbative approach on a background is possible.

It is possible [1] to find Chern-Simons term in $D = 5$ but there is no indication of a perturbative approach to the problem. There is nothing to describe the free theory for gravitation in $D = 5$ as in the case of $D = 3$. It is speculated that "Chern-Simons" in $D = 5$ is self interacting and so would be impossible to write in analogy to the Chern-Simons term in $D = 3$, where is possible to have a part that describes the free theory and another part that describes the gravitational interaction.

For calculation of the propagator in perturbation theory the choice of background is

shown to be important.

Some possible topological terms in $D = 5$ go to zero as the perturbation on a background is introduced. For example in a flat space time background, $\eta_{\mu\nu}$, the bilinear term in the topological Lagrangian, is not possible.

In $D = 3$ we can construct a topological free theory for gravitation on flat space time if we suppose that the field variable $h_{\mu\nu}(x)$ transforms like a tensor.

We wish to do a similar treatment for an Einstein-Hilbert-“Chern-Simons” theory in $D = 5$.

We propose a topological Lagrangian in $D = 5$ similar to the one given in [4]. Since the analysis in $D = 3$ are made with only the first part of the topological Lagrangian with the primary interest being in a free theory; only a part of the topological Lagrangian in $D = 5$ is written here and the coupled Einstein-Hilbert Lagrangian is considered.

The Chern-Simons term in $D = 3$ has a global invariance by diffeomorphism, but the local invariance is guaranteed because $h_{\mu\nu}$ transforms like a tensor.

The global covariance by diffeomorphism for “Chern-Simons” in $D = 5$ is not known. The analogy from first part of Chern-Simons in $D = 3$ is used, but it is assumed that the second term exists and that the local covariance is assumed, since as before $h_{\mu\nu}$ is a tensor.

Assume a topological term in $D = 5$ and a Hartree approximation for our topological term and Chern-Simons in $D = 3$ is considered.

Finally, the calculation of the propagator and an analysis of unitarity in tree level is carried out. The theory is seen as an effective theory. Thus there is no problem with renormalizability.

The Lagrangian for Einstein-Chern-Simons theory in $D = 5$ dimensions is given as

$$\mathcal{L} = \mathcal{L}_{E.H} + \mathcal{L}_{g.f} + \mathcal{L}_{c.s} \quad (1)$$

where $\mathcal{L}_{E.H}$, $\mathcal{L}_{g.f}$ and $\mathcal{L}_{c.s}$ are respectively the Einstein Hilbert Lagrangian, gauge fixing

Lagrangian and the topological Chern-Simons term in $D = 5$ dimensions. These are

$$\mathcal{L}_{E\cdot H} = -\frac{1}{2k^2} \sqrt{-g} R , \quad (2)$$

$$\mathcal{L}_{g\cdot f} = \frac{1}{2\alpha} F_\mu F^\mu , \quad (3)$$

$$\mathcal{L}_{c\cdot h} = \frac{1}{\mu} \varepsilon^{\mu\nu\alpha\beta\gamma} \varepsilon^{\lambda\theta\xi\Delta} \tau \partial_\mu \Gamma_{\lambda\nu}^w \partial_k \Gamma_{\alpha\theta}^k \partial_\psi \Gamma_{\beta\xi}^\psi \partial_w \Gamma_{\gamma\Delta}^\tau . \quad (4)$$

In eq. (2) k is the gravitational constant. R is the usual scalar curvature and $\sqrt{-g}$ is the determinant of the metric.

In equation (3) F_μ represents the De Donder gauge fixing term given by

$$F_\mu [h_{\rho\sigma}] = \partial_\lambda \left(h_\mu^\lambda - \frac{1}{2} \delta_\mu^\lambda h_\nu^\nu \right) \quad (5)$$

and α is the Feynman parameter. In eq. (4) $\varepsilon^{\mu\nu\alpha\beta\gamma}$ and $\Gamma_{\mu\nu}^\alpha$ are Levi-Civita and Christoffel symbols respectively.

The gauge fixing invariance is expressed as

$$\delta h_{\mu\nu}(x) = \partial_\mu \xi_\nu(x) + \partial_\nu \xi_\mu(x) \quad (6)$$

The perturbation theory on flat space time is considered such that

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + kh_{\mu\nu}(x) \quad (7)$$

where $h_{\mu\nu}(x)$ will be the gravitation field variable.

Then eq. (1) has a bilinear form like $h\theta h$, and θ is an operator associated with the spin projection operators in rank-2 tensor space.

The Chern-Simons Lagrangian in $D = 5$ dimension is not of the bilinear form in $h_{\mu\nu}$, but a square bilinear like $hh\theta^*hh$. The Lagrangian can be written as

$$\mathcal{L} = h^{\mu\nu} \theta_{\mu\nu, k\lambda} h^{k\lambda} + h^{w\chi} h^{k\delta} \theta_{w\chi k\delta, \psi\sigma\zeta\tau} h^{\psi\sigma} h^{\zeta\tau} . \quad (8)$$

With this form it is difficult to find the propagator. However use of the Hartree approximation [5] between Chern-Simons in $D = 5$ dimensions and Chern-Simons in $D = 3$ dimensions is possible.

Chern-Simons in $D = 3$ is given by

$$\mathcal{L} = \varepsilon^{\mu\nu\alpha} \left(\Gamma_{\mu\beta}^\lambda \partial_\nu \Gamma_{\alpha\lambda}^\beta + \frac{2}{3} \Gamma_{\mu\sigma}^\rho \Gamma_{\nu\varphi}^\sigma \Gamma_{\alpha\rho}^\varphi \right) \quad (9)$$

If Hartree approximation is assumed it can be shown that

$$\mathcal{L}_{c.h}(D=5) \simeq \lambda^2 \mathcal{L}_{c.h}(D=3)$$

where the left side means the Chern-Simons in $D = 5$, the right side is the Chern-Simons in $D = 3$ dimensions.

The parameter λ^2 is a real parameter and it will describe the physics in the model since some conditions are necessary to achieve unitarity of the theory in the tree level. Essentially what we are doing is to consider the square bilinear $hh\theta^*hh$ as an approximation described by a real parameter times $h\theta h$, where $h\theta h$ is the linearized Chern-Simons in $D = 3$.

Then in the Hartree approximation

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu} (\theta_{\mu\nu,k\lambda} \oplus \theta_{\mu\nu,k\lambda}^*) h^{k\lambda} \quad (10)$$

where $\theta_{\mu\nu,k\lambda}$ is the contribution from the Einstein-Hilbert Lagrangian including the gauge fixing and $\theta_{\mu\nu,k\lambda}^*$ is the operator generated by the topological Chern-Simons term in $D = 3$.

The operators θ and θ^* are given respectively by

$$\theta_{\mu\nu,k\lambda} = \frac{\square}{2} \overset{(2)}{P} + \frac{\square}{2\alpha} \overset{(1)}{P}_m - \square \left(\frac{4\alpha - 3}{4\alpha} \right) \overset{(0)}{P}_s + \frac{\square}{4\alpha} \overset{(0)}{P}_w - \frac{\square\sqrt{3}}{4\alpha} \overset{(0)}{P}_{sw} - \frac{\square\sqrt{3}}{\alpha} \overset{(0)}{P}_{ws} \quad (11)$$

and

$$\theta_{\mu\nu,k\lambda}^* = \frac{4k^2}{\mu} \lambda^2 (S_1 + S_2) . \quad (12)$$

Here $\overset{(2)}{P}$, $\overset{(1)}{P}_m$, $\overset{(0)}{P}_s$, $\overset{(0)}{P}_w$, $\overset{(0)}{P}_{sw}$ and $\overset{(0)}{P}_{ws}$ are spin projection operators.

The two new operators are S_1 and S_2 and are given by.

$$(S_1)_{\mu\nu k\lambda} = \frac{1}{4} (-\square) [\varepsilon_{\mu\alpha\lambda} \partial_k w_\nu^\alpha + \varepsilon_{\mu\nu k} \partial_\lambda w_\nu^\alpha + \varepsilon_{\nu\alpha\lambda} \partial_k w_\mu^\alpha + \varepsilon_{\nu\alpha k} \partial_\lambda w_\mu^\alpha] \quad (13)$$

$$(S_2)_{\mu\nu k\lambda} = \frac{1}{4} \square [\varepsilon_{\mu\alpha\lambda} \eta_{k\nu} + \varepsilon_{\mu\alpha k} \eta_{\lambda\nu} + \varepsilon_{\nu\alpha\lambda} \eta_{k\mu} + \varepsilon_{\nu\alpha k} \eta_{\lambda\mu}] \partial^\alpha . \quad (14)$$

We are looking for the propagator of Einstein Hilbert-Chern-Simons in $D = 5$ dimensions, then we assume a linear combination of the same spin projection operators

$$(\theta^{-1})_{\mu\nu k\lambda} = X \overset{(2)}{P} + Y \overset{(1)}{P}_m + Z \overset{(0)}{P}_s + W \overset{(0)}{P}_w + T \overset{(0)}{P}_{sw} + R \overset{(2)}{P}_{ws} + M S_1 + N S_2 . \quad (15)$$

Now we can calculate the propagator for the field $h_{\mu\nu}(x)$ in $D = 5$ by extending the algebra of Barnes and Rivers [4] and the inverse operator given by eq. (15).

When we take the complete operator from eq. (10) and the inverse operator in eq. (15), the multiplication between them give us the identity in rank-2 tensor space, as

$$\theta_{\mu\nu}^{\rho\sigma} (\theta_{\rho\sigma k\lambda})^{-1} = \left(\overset{(2)}{P} + \overset{(1)}{P}_m + \overset{(0)}{P}_s + \overset{(0)}{P}_w \right)_{\mu\nu, k\lambda} . \quad (16)$$

A system of equation are found from eq. (16) and these are

$$\begin{aligned} \frac{\square}{2} X - \frac{4k^2\lambda^2}{\mu} \square^3 N &= 1 , \\ \frac{\square}{2\alpha} Y &= 1 , \\ \left[\frac{\square}{6} + \frac{\square}{3} \left(\frac{4\alpha - 3}{4\alpha} \right) \right] X + \left[-\frac{\square}{6} - \square \left(\frac{4\alpha - 3}{3\alpha} \right) \right] Z - \frac{\square\sqrt{3}}{4\alpha} R + \frac{2k^2\lambda^2\square^3}{\mu} N &= 1 \\ \left(-\frac{\square}{6} - \frac{\square\sqrt{3}}{3\alpha} \right) T + \frac{\square}{4\alpha} W &= 1 , \\ \square \left(\frac{4\alpha - 3}{3\alpha} \right) T + \frac{\square\sqrt{3}}{4\alpha} W &= 0 , \\ \frac{\square}{4\alpha} R + \frac{\square\sqrt{3}}{12\alpha} X - \frac{\square\sqrt{3}}{3\alpha} Z &= 0 \quad (17) \\ \left(\frac{\square}{2} - \frac{\square}{2\alpha} \right) N + \frac{\square}{2\alpha} M + \frac{4k^2\lambda^2}{\mu} X &= 0 \text{ and} \\ \frac{\square}{2} N + \frac{4k^2\lambda^2}{\mu} X &= 0 , \end{aligned}$$

The coefficients (X, Y, Z, W, T, R, M, N) in the space of coordinates are written as

$$\begin{aligned}
X &= \frac{-2}{-\square - \frac{64\square^2 k^4 \lambda^4}{\mu^2}}, \\
Y &= \frac{2\alpha}{\square}, \\
Z &= \frac{-64k^4 \lambda^4}{64\square k^4 \lambda^4 + \mu^2}, \\
W &= \frac{8(8 - \sqrt{3})(-3 + 4\alpha)}{61\square}, \\
T &= \frac{6(3 - 8\sqrt{3})}{61\square}, \\
R &= \frac{-2(128\square k^4 \lambda^4 + \mu^2)}{\sqrt{3}(64\square^2 k^4 \lambda^4 + \square \mu^2)}, \\
M &= \frac{-16k^2 \lambda^2 \mu}{\square^2(64\square k^4 \lambda^4 + \mu^2)} \text{ and} \\
N &= \frac{-16k^2 \lambda^2 \mu}{\square^2(64\square k^4 \lambda^4 + \mu^2)}.
\end{aligned} \tag{18}$$

The propagator of Einstein-Hilbert-“Chern-Simons” theory in $D = 5$ can be written as

$$\langle h_{k\nu}(x), h_{\mu\lambda}(y) \rangle = i\theta_{\mu\nu, k\lambda}^{-1} \delta^5(x - y) \tag{19}$$

We can define the transition amplitude as

$$\mathcal{A} = \tau^{\mu\nu*}(x) \langle h_{\mu\nu}(x), h_{k\lambda}(y) \rangle \tau^{k\lambda}(y) \tag{20}$$

The coupling between propagator and external currents like energy momentum tensor is compatible with the gauge symmetry eq. (6).

Several coefficients in (18) vanish due to the transversality relation [4]. Only three coefficients survive and are referred as X, Z and N .

These coefficients in momentum space are

$$\begin{aligned}
X &= \frac{-2}{\mathbf{k}^2 \left(1 - \frac{64\mathbf{k}^2 \lambda^4 k^4}{\mu^2}\right)}, \\
Z &= \frac{1}{\mathbf{k}^2 \left(1 - \frac{\mu^2}{64\mathbf{k}^2 k^4 \lambda^4}\right)} \text{ and} \\
N &= \frac{\mu}{(\mathbf{k}^2)^3 \cdot 4k^2 \lambda^2 \left(1 - \frac{\mu^2}{\mathbf{k}^2 k^4 \lambda^4}\right)}.
\end{aligned} \tag{21}$$

In the spin two sector we can see two poles given by

$$\begin{aligned}\mathbf{k}^2 &= 0 \quad \text{and} \\ \mathbf{k}^2 &= \left(\frac{\mu}{8k^2\lambda^2}\right)^2.\end{aligned}\quad (22)$$

In the zero spin sector (Z coefficient) and in the topological sector (N -coefficient we find the same poles.

Observe that when we put $\mu \rightarrow \infty$ we have the Z and N coefficients vanishing and the X coefficient is written as

$$X = -\frac{2}{\mathbf{k}^2}. \quad (23)$$

This means that the dominant term of the propagator in $D = 5$ when the contribution from Chern-Simons term is null is compatible with the result given by [3, 4].

We have pure Einstein theory in $D = 5$ dimensions with a propagator $\langle h, h \rangle \sim \frac{1}{\mathbf{k}^2}$ similar to the Einstein case in $D = 4$.

To verify the unitarity of the theory at tree level eq. (19) is considered in momentum space. The imaginary part of the residues of the amplitude at each pole lead to the necessary unitarity condition.

In momentum space eq. (19) is given by

$$\mathcal{A} = \tau^{\mu\nu*}(\vec{\mathbf{k}}) \langle h_{\mu\nu}(-\vec{\mathbf{k}}), h_{k\lambda}(\vec{\mathbf{k}}) \rangle \tau^{k\lambda}(\vec{\mathbf{k}}) \quad (24)$$

The theory will be free of ghost's if

$$I_m \text{Res} \mathcal{A} \Big|_{\mathbf{k}^2=0} > 0 \quad (25)$$

and

$$I_m \text{Res} \mathcal{A} \Big|_{\mathbf{k}^2=\left(\frac{\mu}{8k^2\lambda^2}\right)^2} > 0. \quad (26)$$

The equations are verified if $|\tau_{k\lambda}|^2 < 0$, and μ or $\lambda^2 < 0$; or if $\frac{\lambda^2}{\mu} < 0$.

The equation (25) will be true if $|\tau_{k\lambda}|^2 < 0$ and $\mu > 0$; or $\lambda^2 > 0$; or $\frac{\lambda^2}{\mu} > 0$.

On taking $\lambda^2 = [-\mu, 0)U(0, +\mu]$, we have two possibilities for propagation of gravitons in the Einstein-“Chern-Simons” theory in $D = 5$. Both poles are dynamical.

The situation here is different from the pure Einstein theory in $D = 4$ and Einstein-Chern-Simons in $D = 3$ dimensions. In pure Einstein theory, $D = 4$ there is only one pole or one massless graviton. The pole $\mathbf{k}^2 = 0$ has propagation in tree level and the Einstein-Hilbert Lagrangian is free of ghost's. For the Einstein-Chern-Simons theory in $D = 3$ we have two poles [3, 4] given by $\mathbf{k}^2 = 0$ and $\mathbf{k}^2 \left(\frac{\mu}{8k^2}\right)^2$, but the dynamics is given by the massive pole. There is no propagation associated with the massless pole.

By taking $\lambda^2 = 1$, there is partial information from Einstein-Chern-Simons in $D = 3$, but it should be emphasized that the propagators in $D = 3$ and $D = 5$ are different.

Finally, if we consider $\lambda^2 = \pm\mu$, in according with the range given above, the pole will be located at $\mathbf{k}^2 = \left(\frac{1}{8k^2}\right)^2$ and the dynamic propagation is guaranteed by unitarity in the tree level. The final result is that the propagation of Einstein-“Chern-Simons” theory in $D = 5$ dimensions is completely determined by the massless graviton, because the gravitational interaction is a large scalar force. The massive pole has a short range for propagating information.

Conclusions:

We have constructed an effective model for Einstein-Chern-Simons theory in $D = 5$ dimensions.

This model has two dynamical poles and the unitarity is analyzed in the tree level. The Lagrangian is free of ghost's and tachyons.

As an objective to treat the problem in perturbative approach, Hartree approximation was used and a convenient topological term like Chern-Simons in $D = 3$ dimensions was constructed unlike the case of pure Einstein in $D = 4$ and the Einstein-Chern-Simons in $D = 3$, here the propagation is associated with both poles. There is no problem with the renormalizability since the theory is an effective model for gravity.

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